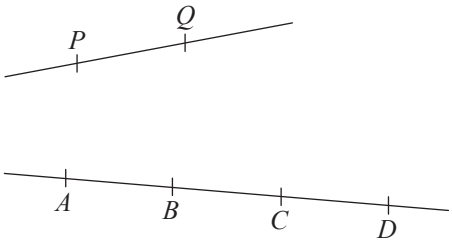
Challenging Problems

1. Draw a circle of radius 6 cm. By drawing smaller circles inside this circle, divide it into 4 regions of equal area.

2. Draw a triangle of any size. By construction partition the triangle into 9 regions of congruent triangles.

3. Triangles are to be drawn so that the vertices are points taken from P, Q, A, B, C and D in the diagram. How many triangles can be drawn?



4. If $x^2 = x - 2$, prove that $x^4 = 2 - 3x$.

5. If $x + y = 3$ and $xy = 1$, calculate

(i) $x^2 + y^2$,

(ii) $x^4 + y^4$,

(iii) $x^8 + y^8$.

6. $a + b + c = 6$

$b + c + d = 9$

$c + d + a = 8$

$a + b + d = 7$

Find the values of a, b, c and d.

7. Solve the equation $(13x + 17)^2 = (11x - 73)^2$.

8. If $a^2x^2 + ax - 1 = 0$, express a in terms of x.

9. Solve the simultaneous equations: $3\left(\frac{x+1}{x-1}\right) + 5\left(\frac{x-1}{x+1}\right) = 8$ and $3\left(\frac{x+1}{x-1}\right) - 5\left(\frac{x-1}{x+1}\right) = 2$.

Adapted:
QUANTUM LEAP • MATHEMATICS – SECONDARY 3/4 (N(A) LEVEL)

10. If you are given the graph of $y = x^2$, how do you use it to solve the equation $x^2 - 3x - 10 = 0$? (You are not required to solve the equation.)

11. The radii, in cm, of 10 spheres are $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5. Calculate the total surface area of the spheres. Leave your answer in terms of π.

12. How long does a train 50 m long and travelling at 30 km/h take to pass completely a train 25 m long and travelling at 60 km/h in the opposite direction?

13. If $\sin A : \sin B : \sin C = 3 : 4 : 5$ and $\angle A + \angle B + \angle C = 180^\circ$, find $\angle A$.

14. Given $S_1 = 1 + 2 + 3 + \ldots + 98 + 99 + 100$ and $S_2 = 100 + 99 + 98 + \ldots + 3 + 2 + 1$, find the sum of all the integers from 1 to 100 inclusive.

15. $\frac{1}{2} + \frac{1}{2} = 1$

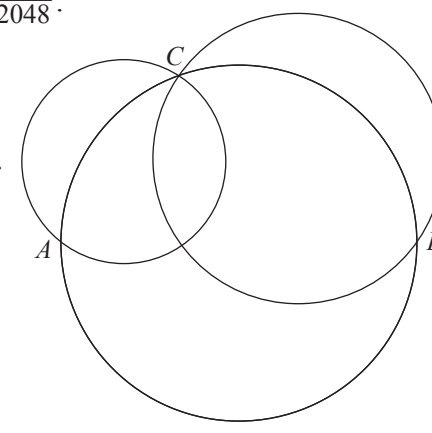
$\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{2} + \frac{1}{2} = 1$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} + \frac{1}{4} + \left(\frac{1}{8} + \frac{1}{8}\right) = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \left(\frac{1}{16} + \frac{1}{16}\right) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 1$

Calculate the sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{512} + \frac{1}{1024} + \frac{1}{2048}$.

16. A, B and C are points on a circle where AB is a diameter. Three circles are drawn with AB, BC and AC as diameters. What can you deduce about the areas of the two smaller circles?



17. Bag A contains 2 white balls, 3 black balls and 5 red balls. Bag B contains 3 white balls, 5 black balls and 2 red balls. Bag C contains 5 white balls, 2 black balls and 3 red balls. A ball is taken from each bag. What is the probability that not all three balls are of the same colour?

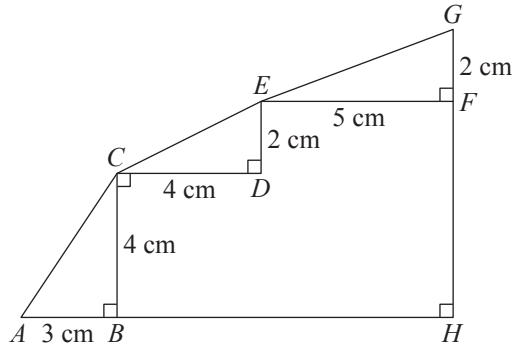
18. Simplify $\dfrac{a^2-(b-c)^2}{(a+b)^2-c^2} \times \dfrac{c^2-(a-b)^2}{(b+c)^2-a^2} \times \dfrac{b^2-(c-a)^2}{(a+c)^2-b^2}$.

19. It is given that $17 + 18 + 19 + \ldots + 94 + 95 + 96 = 4520$.
Use this result to find the sum of $117.5 + 118.5 + 119.5 + \ldots + 194.5 + 195.5 + 196.5$.

20. Solve $(x-1)(x-2)(x-3) = 2(x-1)$.

21. Simplify $\dfrac{a^2+b^2-c^2-2ab}{a^2-b^2-c^2+2bc}$.

22. In the diagram, ABH and GFH are straight lines. Calculate AG.



23. Three dice are thrown. What is the probability that the sum on the three dice is 6?

24. **(i)** Calculate the exact value of $999\ 999^2$.

(ii) Write down the exact value of $999\ 999\ 999^2$.

25. What fraction of the first 100 prime numbers is not divisible by 5?

26. $a : b = \frac{1}{2} : \frac{1}{3}$

$b : c = \frac{1}{4} : \frac{1}{5}$

$c : d = \frac{1}{6} : \frac{1}{7}$

Calculate the ratio $a : b : c : d$.

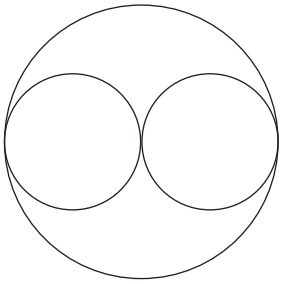
27. If $p + q = 5$ and $pq = 1$, calculate the values of $p - q$.

28. What number is 99 less than 2?

Solutions to:

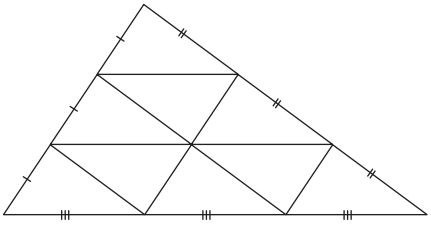
Challenging Problems

1.



Diameter of small circle : Diameter of big circle
= 3 : 6
= 1 : 2
Ratio of their areas is 1 to 4. Area of each of the other two regions
$= \frac{1}{4} \times$ Area of big circle

2.



3. The points taken are

PAB	*QAB*	*PQA*
PAC	*QAC*	*PQB*
PAD	*QAD*	*PQC*
PBC	*QBC*	*PQD*
PBD	*QBD*	
PCD	*QCD*	

16 triangles can be drawn.

4. $x^4 = (x^2)^2$
$= (x-2)^2$
$= x^2 - 4x + 4$
$= (x-2) - 4x + 4$
$= x - 2 - 4x + 4$
$= 2 - 3x$ (proven)

5. **(i)** $x^2 + y^2 = (x+y)^2 - 2xy$
$= 3^2 - 2(1)$
$= 9 - 2$
$= 7$

(ii) $x^4 + y^4 = (x^2 + y^2)^2 - 2(xy)^2$
$= 7^2 - 2(1)^2$
$= 49 - 2$
$= 47$

(iii) $x^8 + y^8 = (x^4 + y^4)^2 - 2(xy)^4$
$= 47^2 - 2(1)^4$
$= 2209 - 2$
$= 2207$

6. Add the LHS and the RHS of the four equations:
$3a + 3b + 3c + 3d = 6 + 9 + 8 + 7$
$3(a + b + c + d) = 30$
$a + b + c + d = 10$

$a + b + c + d = 10$ $\quad$ $a + b + c + d = 10$
$a + b + c = 6$ $\quad$ $b + c + d = 9$
$d = 4$ $\quad$ $a = 1$
$a + b + c + d = 10$ $\quad$ $a + b + c + d = 10$
$a + c + d = 8$ $\quad$ $a + b + d = 7$
$b = 2$ $\quad$ $c = 3$

$\therefore a = 1, b = 2, c = 3, d = 4$

7. $13x + 17 = 11x - 73$ or $13x + 17 = -(11x - 73)$
$2x = -90$ $\quad$ $13x + 17 = -11x + 73$
$x = -45$ $\quad$ $24x = 56$
$x = 2\frac{1}{3}$

$\therefore x = -45$ or $2\frac{1}{3}$

8. $a^2x^2 + ax - 1 = 0$
$x^2a^2 + xa - 1 = 0$

$$a = \frac{-x \pm \sqrt{(x)^2 - 4(x^2)(-1)}}{2x^2}$$
$$= \frac{-x \pm \sqrt{x^2 + 4x^2}}{2x^2}$$
$$= \frac{-x \pm \sqrt{5x^2}}{2x^2}$$
$$= \frac{-x \pm x\sqrt{5}}{2x^2}$$
$$= \frac{-x(1 \pm \sqrt{5})}{2x^2}$$
$$= \frac{-1 \pm \sqrt{5}}{2x}$$

9. Add the LHS and RHS of both equations:

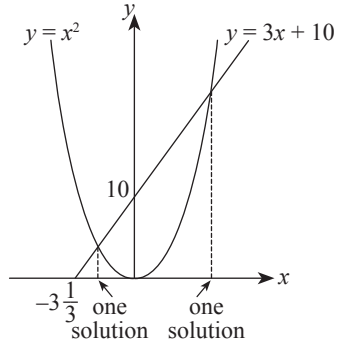
$$6\left(\frac{x+1}{x-1}\right) = 10$$
$$3(x+1) = 5(x-1)$$
$$3x+3 = 5x-5$$
$$8 = 2x$$
$$x = 4$$

10. If $x^2 - 3x - 10 = 0$

$$x^2 = 3x + 10$$

Draw on the graph, the graph of the straight line $y = 3x + 10$.

The root of $x^2 - 3x - 10 = 0$ are the x-coordinates of the intersection of the two graphs.



11. The surface area of the smallest sphere is $4\pi\left(\frac{1}{2}\right)^2 = \pi$ cm^2.

All spheres are similar.

The ratio of their radii is 1 : 2 : 3 : 4 : 5 : 6 : 7 : 8 : 9 : 10.

The ratio of their surface area is 1 : 4 : 9 : 16 : 25 : 36 : 49 : 64 : 81 : 100.

The total surface area $= \pi(1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100)$
$= 385\pi$ cm^2

12. The trains are moving pass each other at (30 + 60) km/h.

(30 + 60) km/h = 90 km/h

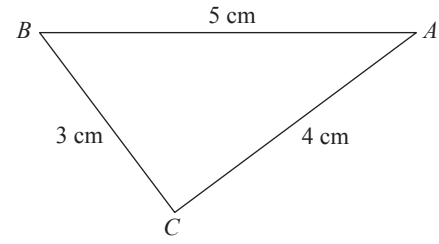
$$= \frac{90 \times 1000 \text{ m}}{3600 \text{ s}}$$
$$= 25 \text{ m/s}$$

Total length of the trains = 50 + 25 = 75 m

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$
$$= \frac{75 \text{ m}}{25 \text{ m/s}}$$
$$= 3 \text{ s}$$

$\therefore$ It takes 3 seconds.

13.



In the triangle,

$$\frac{\sin A}{BC} = \frac{\sin B}{AC} = \frac{\sin C}{AB}$$ sine formula

$$\frac{\sin A}{3} = \frac{\sin B}{4} = \frac{\sin C}{5}$$

$\sin A : \sin B : \sin C = 3 : 4 : 5$

$\angle A + \angle B + \angle C = 180°$ $\angle$ sum of Δ

$$\cos A = \frac{4^2 + 5^2 - 3^2}{2(4)(5)}$$

$A = 36.9°$ (1 d.p.)

14.
$$S_1 = 1 + 2 + 3 + \ldots + 98 + 99 + 100$$
$$S_2 = 100 + 99 + 98 + \ldots + 3 + 2 + 1$$
$$S_1 + S_2 = 101 + 101 + 101 + \ldots + 101 + 101 + 101$$
$$= 101 \times 100$$
$$= 10\,100$$
$$S_1 = S_2$$
$$S_1 = S_2 = \frac{10\,100}{2} = 5050$$

$\therefore 1 + 2 + 3 + \ldots + 98 + 99 + 100 = 5050$

15. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{512} + \frac{1}{1024} + \frac{1}{2048} + \frac{1}{2048}$

$= 1$

$\therefore \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{512} + \frac{1}{1024} + \frac{1}{2048}$

$= 1 - \frac{1}{2048}$

$= \frac{2047}{2048}$

16. $\angle ABC = 90°$ $\angle$ in semi-circle

$AC^2 + BC^2 = AB^2$ Pythagoras' theorem

$$\text{Area of circle} = \pi\left(\frac{\text{diameter}}{2}\right)^2$$
$$= \frac{1}{4}\pi(\text{diameter})^2$$

$\frac{1}{4}\pi AC^2 + \frac{1}{4}\pi BC^2 = \frac{1}{4}\pi AB^2$

$\therefore$ The sum of the areas of the two smaller circles is equal in area to the biggest circle.

17. Probability of getting 3 white balls = Probability of getting a white ball from A × Probability of getting a white ball from B × Probability of getting a white ball from C

$$= \frac{2}{10} \times \frac{3}{10} \times \frac{5}{10}$$
$$= \frac{3}{100}$$
= Probability of getting 3 black balls
= Probability of getting 3 red balls

Probability of getting 3 balls of the same colour $= \frac{3}{100} \times 3 = \frac{9}{100}$

Probability of not getting 3 balls of the same colour $= 1 - \frac{9}{100} = \frac{91}{100}$

18. $\dfrac{a^2-(b-c)^2}{(a+b)^2-c^2} \times \dfrac{c^2-(a-b)^2}{(b+c)^2-a^2} \times \dfrac{b^2-(c-a)^2}{(a+c)^2-b^2}$

$$= \frac{[a+(b-c)][a-(b-c)][c+(a-b)][c-(a-b)][b+(c-a)][b-(c-a)]}{[(a+b)+c][(a+b)-c][(b+c)+a][(b+c)-a][(a+c)+b][(a+c)-b]}$$
$$= \frac{(a+b-c)(a-b+c)(c+a-b)(c-a+b)(b+c-a)(b-c+a)}{(a+b+c)(a+b-c)(b+c+a)(b+c-a)(a+c+b)(a+c-b)}$$
$$= \frac{(a+c-b)(b+c-a)(b-c+a)}{(a+b+c)^3}$$

19. From 17 to 96 there are 96 – 16 or 80 terms
117.5 + 118.5 + 119.5 + … + 194.5 + 195.5 + 196.5
= (100.5 + 17) + (100.5 + 18) + (100.5 + 19) + … + (100.5 + 94) + (100.5 + 95) + (100.5 + 96)
= (100.5 + 100.5 + 100.5 + … + 100.5 + 100.5 + 100.5) + (17 + 18 + 19 + … + 94 + 95 + 96)
= 100.5 × 80 + 4520
= 12560

20.
$$(x-1)(x-2)(x-3) = 2(x-1)$$
$$(x-1)(x-2)(x-3) - 2(x-1) = 0$$
$$(x-1)[(x-2)(x-3) - 2] = 0$$
$$(x-1)(x^2-5x+4) = 0$$
$$(x-1)(x-1)(x-4) = 0$$
$$(x-1)^2(x-4) = 0$$
$$\therefore x = 1 \text{ or } 4$$

21. $\dfrac{a^2+b^2-c^2-2ab}{a^2-b^2-c^2+2bc} = \dfrac{(a^2+b^2-2ab)-c^2}{a^2-(b^2+c^2-2bc)}$

$$= \frac{(a-b)^2-c^2}{a^2-(b-c)^2}$$
$$= \frac{(a-b+c)(a-b-c)}{[a+(b-c)][a-(b-c)]}$$
$$= \frac{(a-b+c)(a-b-c)}{(a+b-c)(a-b+c)}$$
$$= \frac{a-b-c}{a+b-c}$$

22.
$AH = AB + CD + EF$
$= 3 + 4 + 5$
$= 12$ cm
$HG = 4 + 2 + 2$
$= 8$ cm
$AG^2 = AH^2 + HG^2$ Pythagoras' theorem
$= 12^2 + 8^2$
$= 208$ cm^2
$AG = \sqrt{208}$
$= 14.4$ cm (3 s.f.)

23. Three dice can be thrown in 6^3 or 216 ways.
A sum of 6 can be obtained in the following 10 ways:

1, 1, 4	1, 4, 1	4, 1, 1
1, 2, 3	1, 3, 2	2, 1, 3
2, 3, 1	3, 1, 2	3, 2, 1
2, 2, 2		

The probability of throwing a sum of 6 with 3 dice
$= \frac{10}{216} = \frac{5}{108}$

24. (i) $999\,999^2 = 999\,999 \times 999\,999$
$= 999\,999 \times (1000\,000 - 1)$
$= 999\,999\,000\,000 - 999\,999$
$= 999\,998\,000\,001$

(ii) $999\,999\,999^2 = 999\,999\,998\,000\,000\,001$

25. The only prime number that is divisible by 5 is 5.
99 of the first hundred prime number is not divisible by 5.
The fraction is $\frac{99}{100}$.

26. $a : b = \frac{1}{2} : \frac{1}{3} = 3 : 2$

$b : c = \frac{1}{4} : \frac{1}{5} = 5 : 4$

$c : d = \frac{1}{6} : \frac{1}{7} = 7 : 6$

a	:	b	:	c		a	:	b	:	c	:	d
3	:	2				15	:	10	:	8		
		5	:	4						7	:	6
15	:	10	:	8		105	:	70	:	56	:	48

27. $(p + q)^2 = p^2 + 2pq + q^2$

$(p + q)^2 - 2pq = p^2 + q^2$

$p^2 + q^2 = 5^2 - 2(1)$

$= 23$

$(p - q)^2 = p^2 - 2pq + q^2$

$= 23 - 2(1)$

$= 21$

$p - q = \pm\sqrt{21}$

28. $2 - 99 = -97$. The number is -97.