Congruent and Similar Triangles

8.1 Tests for Congruency between Two Triangles

(i) **SSS** Property
The three sides of one triangle are equal to the three sides of the other triangle.

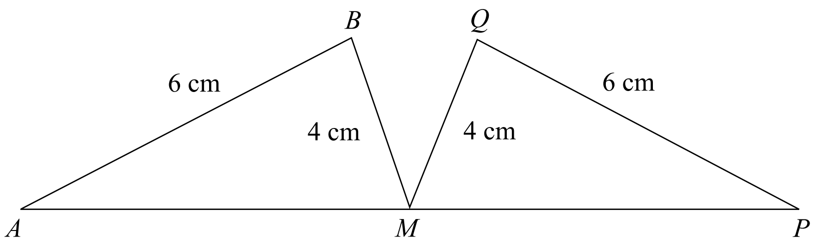
(ii) **SAS** Property
The two sides and the included angle of one triangle are equal to the two sides and the included angle of the other triangle.

(iii) **AAS** Property
The two angles and a side of one triangle are equal to the two angles and a corresponding side of the other triangle.

(iv) **RHS** Property
The hypotenuse and one side of one right-angled triangle are equal to the hypotenuse and one side of the other right-angled triangle.

Example 1

In the diagram, AP is a straight line such that M is the midpoint of AP.



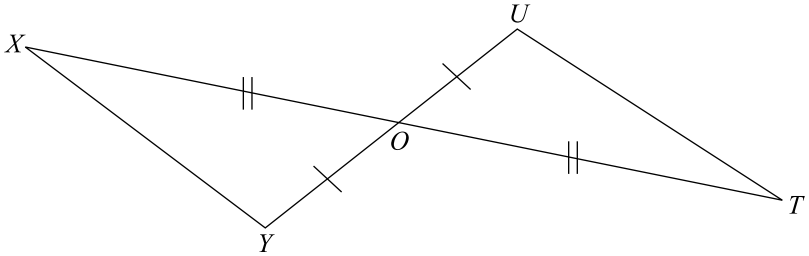
(a) Prove that ΔABM and ΔPQM are congruent.
(b) Hence, write down all the corresponding angles in the two triangles.

Solution: (a) In ΔABM and ΔPQM,
$AB = PQ = 6$ cm
$BM = QM = 4$ cm
$AM = PM$ (M is the midpoint of AP)
$\therefore \Delta ABM \equiv \Delta PQM$ (SSS)

(b) Since $\Delta ABM \equiv \Delta PQM$,
$\angle ABM = \angle PQM$
$\angle BAM = \angle QPM$
$\angle AMB = \angle PMQ$

Example 2

In the diagram, XT and YU are straight lines.



(a) Give a reason why the pair of triangles is congruent.

(b) Given that $\angle OXY = 38°$ and $\angle TOU = 50°$, state the value of $\angle OUT$.

Solution: (a)

$OX = OT$	(given)
$\angle XOY = \angle TOU$	(vert. opp. $\angle$s)
$OY = OU$	(given)
$\therefore \Delta OXY \equiv \Delta OTU$	(SAS)

(b) Since $\Delta OXY \equiv \Delta OTU$,

$\angle OTU = \angle OXY$

$= 38°$

$\angle OUT = 180° - \angle TOU - \angle OTU$ (sum $\angle$s in Δ)

$= 180° - 50° - 38°$

$= 92°$

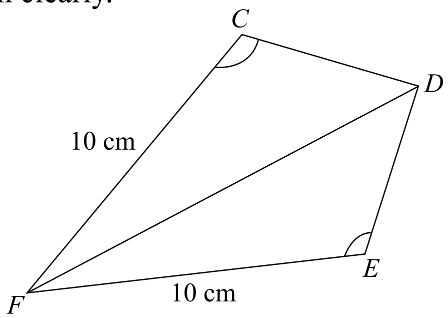
Note

The angle in the SAS rule must be the included angle. Otherwise, the SAS rule is not applicable.

In the example above, the included angles are $\angle XOY$ and $\angle TOU$ but not the other angles.

Example 3

In the diagram below, $\angle C = \angle E$ and $CF = EF = 10$ cm. Is the pair of triangles congruent? Give your reason clearly.



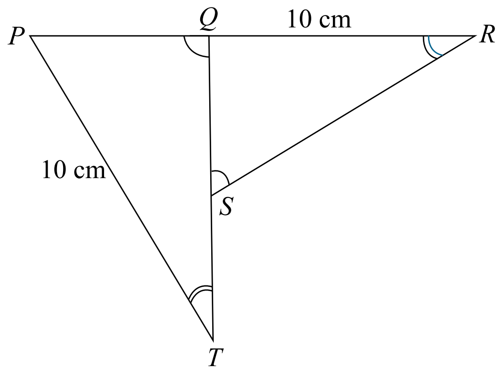
Solution: Although $CF = EF = 10$ cm (given)
$\angle DCF = \angle DEF$ (given)
DF is common
but $\angle DCF$ and $\angle DEF$ are <u>not included angles</u>.
$\therefore$ ΔCDF and ΔEDF <u>cannot</u> be concluded as congruent.

Note

The angle in the SAS rule must be <u>the included angle</u>. Otherwise, the SAS rule is not applicable.
In the example above, the included angles are <u>∠CFD and ∠EFD</u> which are <u>not</u> given.

Example 4

In the diagram, $PT = QR = 10$ cm, $\angle Q = \angle S$ and $\angle T = \angle R$.
(a) Name, by giving your reason, a pair of congruent triangles.
(b) Write down the other equal angles and equal pair of sides.



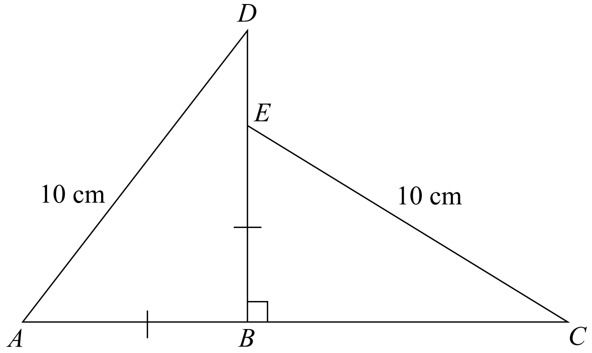
Solution: (a) $\angle PQT = \angle QSR$ (given)
$\angle PTQ = \angle QRS$ (given)
$PT = QR = 10$ cm (given)
$\therefore \Delta PQT \equiv \Delta QSR$ (AAS)

(b) Since $\Delta PQT \equiv \Delta QSR$,
$\angle QPT = \angle SQR$
$PQ = QS$
$QT = SR$

Example

5 In the diagram, ABC is a straight line, $AB = BE = 6$ cm, $AD = CE = 10$ cm and $\angle EBC = 90°$.

(a) Show that ΔABD is congruent to ΔEBC.
(b) Calculate the length of ED.
(c) Find the exact value of $\cos \angle DEC$.



Solution: (a) $\angle ABD = \angle EBC = 90°$ (given)
$AD = EC = 10$ cm (given)
$AB = EB = 6$ cm (given)
$\therefore \Delta ABD \equiv \Delta EBC$ (RHS)

(b) $BD^2 = 10^2 - 6^2$

$BD = \sqrt{64}$
$= 8$ cm

$DE = BD - BE$
$= 8 - 6$
$= 2$ cm

(c) $\cos \angle DEC = -\cos \angle BEC$

$$= -\frac{EB}{EC}$$

$$= -\frac{6}{10}$$

$$= -\frac{3}{5}$$

8.2 Tests for Similarity between Two Triangles

(i) The <u>two angles</u> of one triangle are equal to <u>two angles</u> of the other triangle.

(ii) The <u>three corresponding sides</u> of the two triangles are proportional.

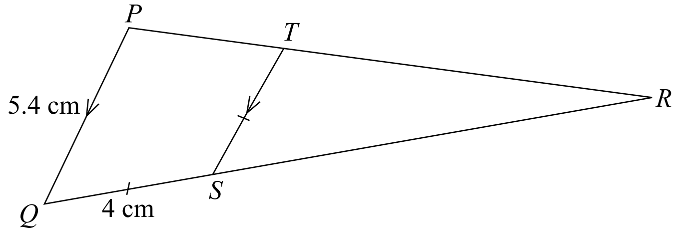
(iii) The <u>two corresponding sides</u> of the two triangles are proportional **<u>and</u>** the <u>included angle</u> of one triangle is equal to the <u>included</u> angle of the other triangle.

Example 6

In the diagram, QSR is a straight line, PQ is parallel to TS, $PQ = 5.4$ cm and $TS = SQ = 4$ cm.

(a) Show that ΔPQR and ΔTSR are similar.

(b) Find the length of RS.



Solution: (a) $\angle QPR = \angle STR$ (corr. $\angle$s, $PQ//TS$)

$\angle PQR = \angle TSR$ (corr. $\angle$s, $PQ//TS$)

As the two angles of the triangles are equal, ΔPQR and ΔTSR are similar.

(b) Since ΔPQR and ΔTSR are similar,

$$\frac{RS}{RQ} = \frac{TS}{PQ}$$

$$\frac{RS}{RS + 4} = \frac{4}{5.4}$$

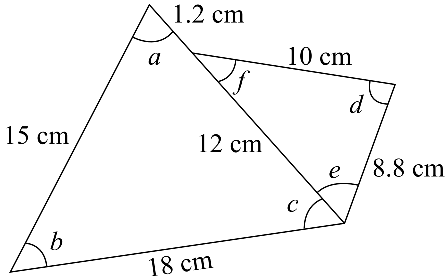
$$5.4RS = 4RS + 16$$

$$1.4RS = 16$$

$$RS = 11\frac{3}{7} \text{ cm}$$

Example 7

(a) Show that the two triangles are similar.
(b) Write down the three pairs of angles that are equal.



Solution: (a) Consider the sides of two triangles:

$$\frac{12}{18} = \frac{2}{3}, \quad \frac{10}{15} = \frac{2}{3} \quad \text{and} \quad \frac{8.8}{12 + 1.2} = \frac{2}{3}$$

Since $\frac{12}{18} = \frac{10}{15} = \frac{8.8}{13.2} = \frac{2}{3}$, the two triangles are similar.

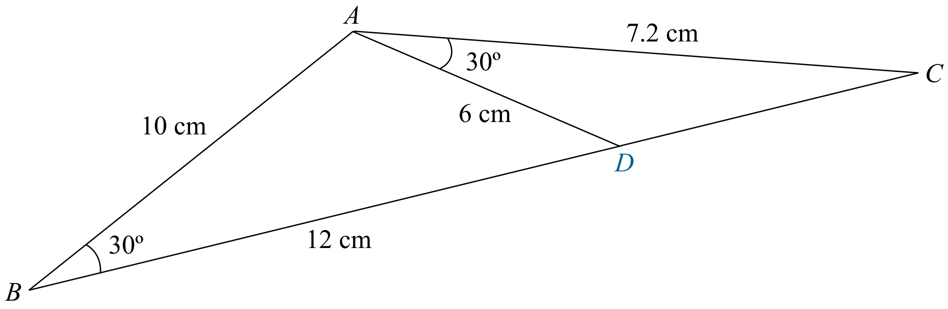
(b) $\angle a = \angle d$ ($\angle a$ is between sides of 15 cm and 13.2 cm, $\angle d$ is between sides of 10 cm and 8.8 cm)

$\angle b = \angle f$ ($\angle b$ is between sides of 15 cm and 18 cm, $\angle f$ is between sides of 10 cm and 12 cm)

$\angle c = \angle e$ ($\angle c$ is between sides of 13.2 cm and 18 cm, $\angle e$ is between sides of 8.8 cm and 12 cm)

Example 8

(a) Name a pair of triangles that are similar. Explain your answer.
(b) Find the length of CD.



Solution: (a) $\frac{AB}{DA} = \frac{10}{6} = \frac{5}{3}$

$\frac{DB}{CA} = \frac{12}{7.2} = \frac{5}{3}$

$\angle ABD = \angle DAC = 30°$ (given)

As the two corresponding sides of the triangles are proportional and their included angles are equal, ΔABD and ΔDAC are similar.

(b) Since ΔABD and ΔDAC are similar, $\frac{CD}{DA} = \frac{DA}{AB}$

$\frac{CD}{6} = \frac{6}{10}$

$CD = 3.6$ cm