Trigonometry and Pythagoras' Theorem

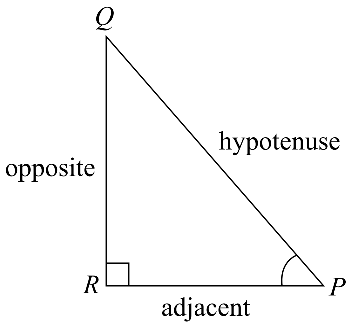
Trigonometrical Ratio of Acute Angles

For the right-angled triangle PQR,

(i) $\tan \angle P = \dfrac{\text{opposite side (opp)}}{\text{adjacent (adj)}} = \dfrac{QR}{PR}$, $\Rightarrow$ TOA

(ii) $\cos \angle P = \dfrac{\text{adjacent side (adj)}}{\text{hypotenuse (hyp)}} = \dfrac{PR}{PQ}$, $\Rightarrow$ CAH

(iii) $\sin \angle P = \dfrac{\text{opposite side (opp)}}{\text{hypotenuse (hyp)}} = \dfrac{QR}{PQ}$. $\Rightarrow$ SOH

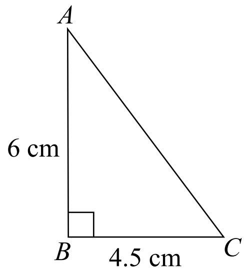


Tips *The three trigonometrical ratios are only applicable to any right-angled triangle.*

Example

1 In the diagram, triangle ABC is a right-angled triangle.
Find the exact numerical value of

(a) $\sin \angle A$,
(b) $\cos \angle A$,
(c) $\tan \angle A$,
(d) $\sin \angle C$,
(e) $\cos \angle C$,
(f) $\tan \angle C$.



Solution: $AC = \sqrt{6^2 + 4.5^2}$
$= 7.5$ cm
(For $\angle A$, BC is the opposite side, AB is the adjacent side and AC is the hypotenuse.)

(a) $\sin \angle A = \dfrac{BC}{AC}$ (using SOH)
$= \dfrac{4.5}{7.5}$
$= \dfrac{3}{5}$

(b) $\cos \angle A = \dfrac{AB}{AC}$ (using CAH)
$= \dfrac{6}{7.5}$
$= \dfrac{4}{5}$

(c) $\tan \angle A = \dfrac{BC}{AB}$ (using TOA)

$= \dfrac{4.5}{6}$

$= \dfrac{3}{4}$

(For $\angle C$, BC is the adjacent side, AB is the opposite side and AC is the hypotenuse.)

(d) $\sin \angle C = \dfrac{AB}{AC}$ (using SOH)

$= \dfrac{6}{7.5}$

$= \dfrac{4}{5}$

(e) $\cos \angle C = \dfrac{BC}{AC}$ (using CAH)

$= \dfrac{4.5}{7.5}$

$= \dfrac{3}{5}$

(f) $\tan \angle C = \dfrac{AB}{BC}$ (using TOA)

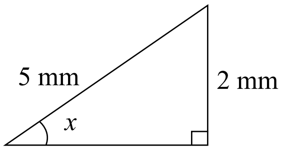
$= \dfrac{6}{4.5}$

$= \dfrac{4}{3}$

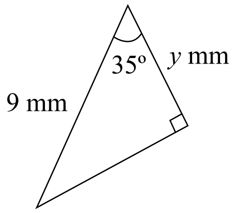
Example 2

Solve each of the following unknowns in the following diagrams.

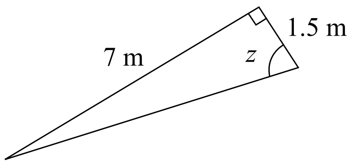
(a)



(b)



(c)



Solution: (a) Refer to $\angle x$. The two known sides are the opposite sides of 2 mm and the hypotenuse of 5 mm.

$\sin x = \dfrac{2}{5}$

$x = \sin^{-1} \dfrac{2}{5}$

$= 23.6^\circ$ (1 d.p.)

[$\angle x$ is an acute angle as shown in the diagram, $x = 180^\circ - 23.6^\circ = 156.4^\circ$ is ***not applicable***.]

(b) Refer to the angle that is equal to 35 degrees. The two known sides are the adjacent sides of y mm and the hypotenuse of 9 mm.

$$\cos 35^\circ = \frac{y}{9}$$
$$y = 9 \cos 35^\circ$$
$$= 7.37 \text{ mm} \qquad (3 \text{ s.f.})$$

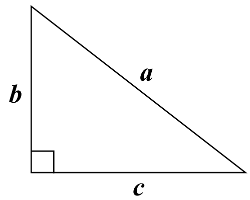
(c) Refer to $\angle z$. The two known sides are the opposite sides of 7 m and the adjacent side of 1.5 m.

$$\tan z = \frac{7}{1.5}$$
$$z = \tan^{-1} \frac{7}{1.5}$$
$$= 77.9^\circ \qquad (1 \text{ d.p.})$$

Pythagoras' Theorem and Converse of Pythagoras' Theorem

Pythagoras' Theorem

In a ***right-angled*** triangle with sides $\boldsymbol{b}$, $\boldsymbol{c}$ and hypotenuse $\boldsymbol{a}$, ***the square of the hypotenuse is equal to the sum of the squares of the other two sides***.



$$a^2 = b^2 + c^2$$

If the lengths of any two sides are known, the length of the third side can be found.

Converse of Pythagoras' Theorem

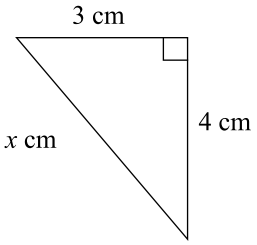
In a triangle with sides $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$,

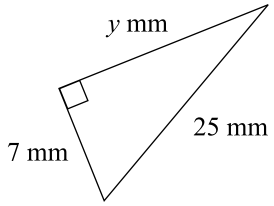
if $\quad b^2 + c^2 = a^2 \quad$ holds,

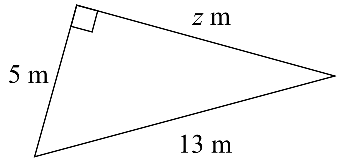
then the triangle is a ***right-angled*** triangle and the angle facing side $\boldsymbol{a}$ (hypotenuse) is a right angle.

Example 3

Calculate each of the following unknowns:

(a) 

(b) 

(c) 

Solution: (a) The side with length x cm (opposite the right angle) is the hypotenuse.

$$x^2 = 3^2 + 4^2$$
$$= 9 + 16$$
$$= 25$$
$$x = \sqrt{25} \qquad (x > 0)$$
$$= 5$$

(b) The side with length 25 mm is the hypotenuse.

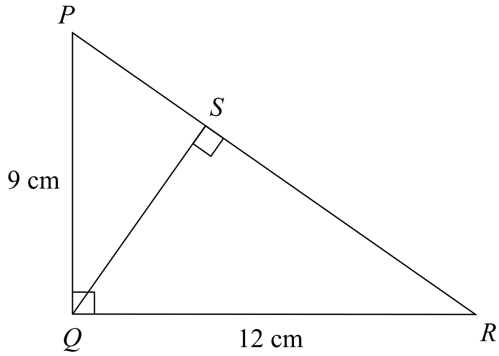
$$y^2 + 7^2 = 25^2$$
$$y^2 = 25^2 - 7^2$$
$$= 625 - 49$$
$$= 576$$
$$y = \sqrt{576} \qquad (y > 0)$$
$$= 24$$

(c) The side with length 13 m is the hypotenuse.

$$z^2 + 5^2 = 13^2$$
$$z^2 = 13^2 - 5^2$$
$$= 169 - 25$$
$$= 144$$
$$z = \sqrt{144} \qquad (z > 0)$$
$$= 12$$

Example 4

The figure below shows a triangle PQR and a straight line QS. Given that $3PS = 2SR$, $PQ = 9$ cm, $QR = 12$ cm and $\angle PQR = \angle PSQ = 90^\circ$, find the length of QS. Give your answer correct to three significant figures.



Solution: Considering $\triangle PQR$,

$$
\begin{aligned}
PR^2 &= PQ^2 + QR^2 \\
&= 9^2 + 12^2 \\
&= 225 \\
PR &= \sqrt{225} \qquad (PR > 0) \\
&= 15 \text{ cm}
\end{aligned}
$$

Considering $\triangle PQS$,

$$
\begin{aligned}
PS &= \frac{2}{5}PR \qquad (3PS = 2SR) \\
&= \frac{2}{5}(15) \\
&= 6 \text{ cm} \\
PQ^2 &= PS^2 + QS^2 \\
9^2 &= 6^2 + QS^2 \\
QS^2 &= 81 - 36 \\
&= 45 \\
QS &= \sqrt{45} \qquad (QS > 0) \\
&= 6.71 \text{ cm} \qquad \text{(3 s.f.)}
\end{aligned}
$$