Coordinate Geometry

Example 1

The origin O is (0, 0), S is (3, 2) and T is (–5, 1).
Find
(a) the coordinates of the midpoint of OS,
(b) the coordinates of R such that $ORST$ is a parallelogram.

Solution: (a) Midpoint of $OS = \left(\frac{0+3}{2}, \frac{0+2}{2}\right)$
$= (1.5, 1)$

(b) Let R be (a, b).
Midpoint of $RT = \left(\frac{a+(-5)}{2}, \frac{b+1}{2}\right)$
Since $ORST$ is a parallelogram, midpoint of RT = midpoint of OS.
$\left(\frac{a+(-5)}{2}, \frac{b+1}{2}\right) = (1.5, 1)$

$\frac{a+(-5)}{2} = 1.5$ $\qquad$ $\frac{b+1}{2} = 1$
$a - 5 = 3$ $\qquad$ $b + 1 = 2$
$a = 8$ $\qquad$ $b = 1$
$\therefore R(8, 1)$

Example 2

The coordinates of three points, A, B and C, are (7, 2), (4, –1) and (5, 0) respectively. Find the coordinates of D if $ABCD$ is a parallelogram.

Solution: (a) Let D be (x, y).
Since $ABCD$ is a parallelogram, midpoint of BD = midpoint of AC.
$\left(\frac{x+4}{2}, \frac{y+(-1)}{2}\right) = \left(\frac{7+5}{2}, \frac{2+0}{2}\right)$

$\frac{x+4}{2} = \frac{7+5}{2}$ $\qquad$ $\frac{y+(-1)}{2} = \frac{2+0}{2}$
$x + 4 = 7 + 5$ $\qquad$ $y + (-1) = 2 + 0$
$x = 8$ $\qquad$ $y = 3$
$\therefore D(8, 3)$

Example 3

The equation of the line KL is $3y = x - 6$. Given that the line PQ is parallel to the line KL and passes through the origin, find
(a) the coordinates of the point which the line KL cuts at the y-axis,
(b) the equation of the line PQ.

Solution: (a) $3y = x - 6$

At y-axis, $x = 0$

$3y = -6$

$y = -2$

$\therefore (0, -2)$

(b) $3y = x - 6$

$y = \frac{1}{3}x - 2$

$PQ \,//\, KL$,

Gradient of PQ = gradient of KL

$= \frac{1}{3}$

Equation of PQ: $\frac{y - 0}{x - 0} = \frac{1}{3}$

$3y = x$

Example 4

The points U and V are (2, 3) and $(a, a - 1)$ respectively.
(a) Write down, in terms of a, the gradient of the line UV.
(b) Given that the line UV is parallel to the x-axis, find the value of a.

Solution: (a) Gradient $= \frac{a - 1 - 3}{a - 2}$

$= \frac{a - 4}{a - 2}$

(b) Line $UV \,//\, x$-axis,

Gradient of $UV = 0$

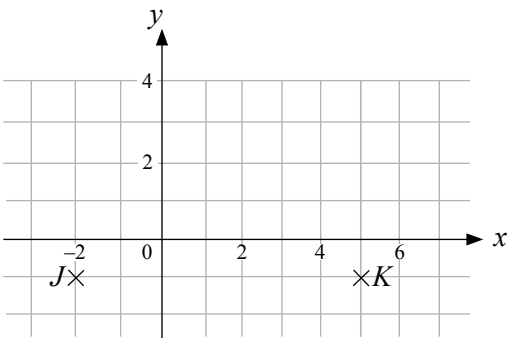
$\frac{a - 4}{a - 2} = 0$

$a - 4 = 0$

$a = 4$

Example 5

JKL is an isosceles triangle such that $JL = KL$.
Given J is $(-2, -1)$, K is $(5, -1)$ and L is (m, n).



(a) Find the equation of line JK.
(b) Write down the value of m.
(c) The area of ΔJKL is 7 units^2. Find the two possible values of n.

Solution: (a) JK is a horizontal line (// x-axis)
$\therefore$ equation of line LK: $y = -1$

(b) Since $JL = KL$,
x-coordinate of L
= mid-value of x-coordinates of J and K

$$= \frac{-2 + 5}{2}$$
$$= 1.5$$
$\therefore m = 1.5$

(c) Area of $\triangle JKL$ = 7 units

$$\frac{1}{2} \times JK \times \perp \text{height} = 7$$
$$\frac{1}{2} \times [5 - (-2)] \times \perp \text{height} = 7$$
$$\perp \text{height} = 2$$
$$|n - (-1)| = 2$$
$$|n + 1| = 2$$
$$n + 1 = \pm 2$$
$$n = -3 \text{ or } 1$$

Example 6

The coordinates of three points, P, Q and R, are (6, 0), (4, 4) and (–2, 1) respectively.

(a) Calculate the length of the line segment
  (i) PQ,
  (ii) QR,
  (iii) PR.
  Leave your answers in the form of $\sqrt{a}$ when necessary.

(b) Hence,
  (i) show that ΔPQR is a right-angled triangle.
  (ii) Calculate the area of ΔPQR.

Solution: (a) (i) $PQ = \sqrt{(6-4)^2 + (0-4)^2}$
$= \sqrt{4 + 16}$
$= \sqrt{20}$
$= 4.47$ units (3 sig.fig.)

(ii) $QR = \sqrt{(-2-4)^2 + (1-4)^2}$
$= \sqrt{36 + 9}$
$= \sqrt{45}$
$= 6.71$ units (3 sig. fig.)

(iii) $PR = \sqrt{(-2-6)^2 + (1-0)^2}$
$= \sqrt{64 + 1}$
$= \sqrt{65}$
$= 8.06$ units (3 sig.fig.)

(b) (i) $(\sqrt{20})^2 + (\sqrt{45})^2 = (\sqrt{65})^2$
$PQ^2 + QR^2 = PR^2$
$\therefore$ by converse of Pythagoras' Theorem,
$\triangle PQR$ is a right-angled triangle.

(ii) $\triangle PQR$ is a right-angled $\triangle$ such that $\angle Q = 90^\circ$,
$\therefore$ area of $\triangle PQR = \frac{1}{2} \times PQ \times QR$
$= \frac{1}{2} \times \text{R20} \times \text{R45}$
$= 15$ units^2

Example 7

Four points, P, Q, R and S, are (5, 0), (2, 0), (0, 4) and (m, n) respectively. Given $PQRS$ is a trapezium such that PQ is parallel to RS and the area of the trapezium is 24 units^2.

(a) Find the length of the line segment PQ.

(b) Write down the value of n.

(c) Find the value of m, where $m > 0$.

Solution: (a) $PQ = |5 - 2|$ (PQ is a horizontal line)

$= 3$ units

(b) Since PQ // RS, RS is also a horizontal line.

$\therefore$ y-coordinate of S = y-coordinare of R

$= 4$

$\therefore$ $n = 4$

(c) Area of trapezium $= \frac{1}{2} \times (a + b) \times h$

h = height

= difference in y-coordinates

$= 4 - 0$

$= 4$ units

$a = PQ = 3$ units

$b = |m - 0| = |m|$ units

$\frac{1}{2} \times (3 + |m|) \times 4 = 24$

$3 + |m| = 12$

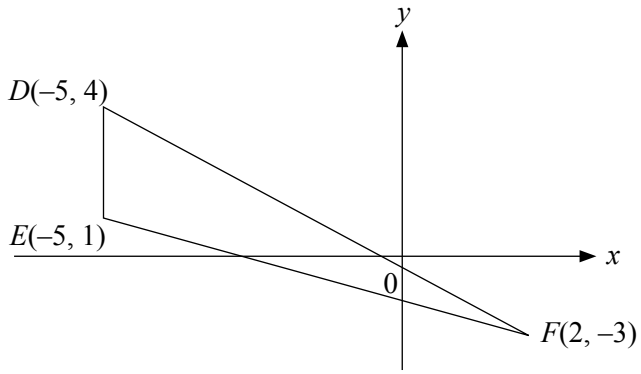
$|m| = 9$

$m = \pm 9$

$m = 9$ ($m > 0$)

Example 8

D, E and F are the points $(-5, 4)$, $(-5, 1)$ and $(2, -3)$ as shown in the diagram.



Find

(a) the length of the line segment DE,

(b) the gradient of the line DF,

(c) the equation of the line which passes through the origin and parallel to DF,

(d) the area of triangle DEF,

(e) the shortest distance from point E to the line DF.

Solution: (a) Length $DE = |1 - 4|$ (vertical line)

$= |-3|$

$= 3$ units

(b) Gradient of $DF = \dfrac{4 - (-3)}{-5 - 2}$

$= -1$

(c) Equation of line: $\dfrac{y - 0}{x - 0} = -1$

$y = -x$

(d) Base $= DE = 3$ units

Height = difference in x-coordinates

$= |2 - (-5)|$

$= 7$ units

Area of $\triangle DEF = \dfrac{1}{2} \times DE \times$ height

$= \dfrac{1}{2} \times 3 \times 7$

$= 10.5$ units^2

(e) Length $DF = \sqrt{(-5 - 2)^2 + [4 - (-3)]^2}$

$= \sqrt{49 + 49}$

$= \sqrt{98}$ units

Area of $\triangle DEF = \dfrac{1}{2} \times DF \times$ shortest distance

$\dfrac{1}{2} \times \sqrt{98} \times$ shortest distance $= 10.5$ units^2

shortest distance $= \dfrac{10.5 \times 2}{\sqrt{98}}$

$= 2.12$ units (3 sig.fig.)