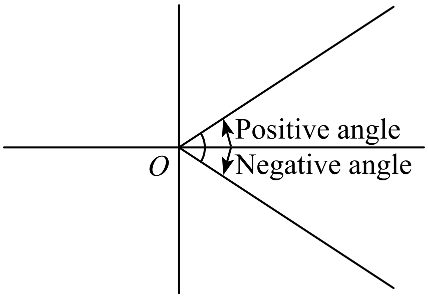
Trigonometric Functions, Identities and Equations

Key Concepts

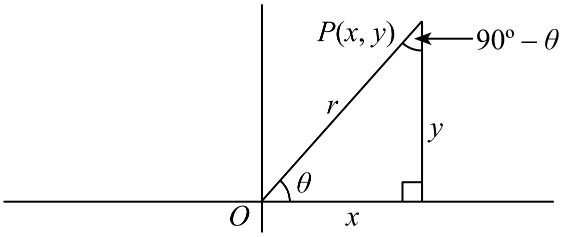
Positive and Negative Angles

1 A positive angle is an anticlockwise rotation from the positive x-axis about the origin. A negative angle is a clockwise rotation from the positive x-axis about the origin.



Trigonometric Ratios of Acute Angles and Complementary Angles

2



Acute angles

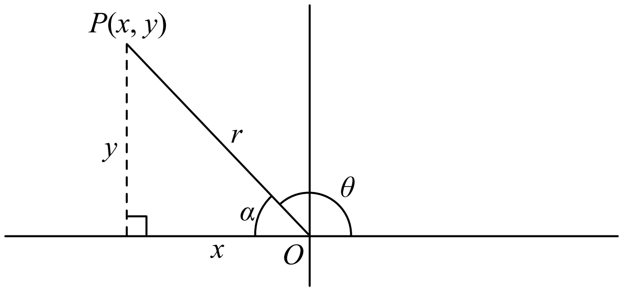
$\tan\theta = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{y}{x}$	$\cot\theta = \dfrac{\text{adjacent side}}{\text{opposite side}} = \dfrac{x}{y} = \dfrac{1}{\tan\theta}$
$\cos\theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{x}{r}$	$\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent side}} = \dfrac{r}{x} = \dfrac{1}{\cos\theta}$
$\sin\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{y}{r}$	$\text{cosec}\,\theta = \dfrac{\text{hypotenuse}}{\text{opposite side}} = \dfrac{r}{y} = \dfrac{1}{\sin\theta}$

Complementary angles

$\sin(90° - \theta) = \cos\theta$	$\cos(90° - \theta) = \sin\theta$
$\tan(90° - \theta) = \cot\theta$	$\cot(90° - \theta) = \tan\theta$
$\sec(90° - \theta) = \text{cosec}\,\theta$	$\text{cosec}(90° - \theta) = \sec\theta$

Trigonometric Ratios of All Angles

3

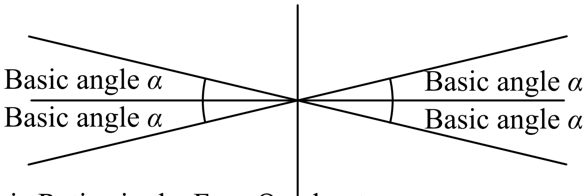


All Angles

$\tan\theta = \dfrac{y\text{-coordinate of } P}{x\text{-coordinate of } P} = \dfrac{y}{x}$	$\cot\theta = \dfrac{y\text{-coordinate of } P}{x\text{-coordinate of } P} = \dfrac{x}{y} = \dfrac{1}{\tan\theta}$
$\cos\theta = \dfrac{x\text{-coordinate of } P}{OP} = \dfrac{x}{r}$	$\sec\theta = \dfrac{OP}{x\text{-coordinate of } P} = \dfrac{r}{x} = \dfrac{1}{\cos\theta}$
$\sin\theta = \dfrac{y\text{-coordinate of } P}{OP} = \dfrac{y}{r}$	$\operatorname{cosec}\theta = \dfrac{OP}{y\text{-coordinate of } P} = \dfrac{r}{y} = \dfrac{1}{\sin\theta}$

Basic Angle (Associated Acute Angle or Reference Angle)

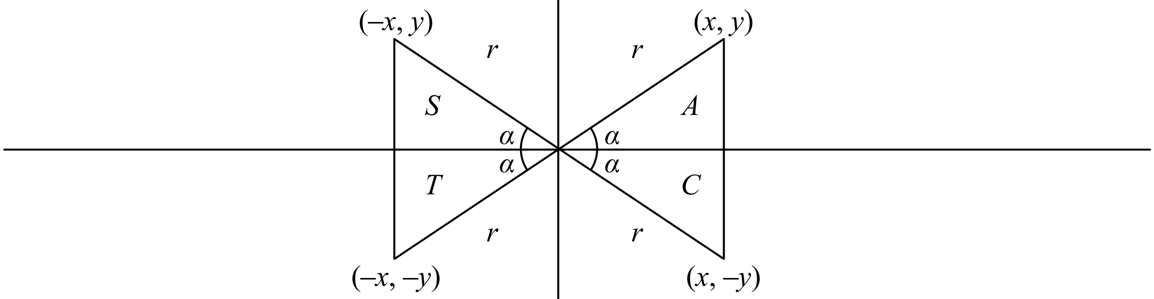
4 The basic angle α is the acute angle between a rotating radius and the x-axis.



5 Signs of Trigonometric Ratios in the Four Quadrants

Obtuse Angles (Science)
$\sin(180° - \alpha) = +\sin\alpha$
$\cos(180° - \alpha) = -\cos\alpha$
$\tan(180° - \alpha) = -\tan\alpha$

Acute Angles (All)
$+\sin\alpha$
$+\cos\alpha$
$+\tan\alpha$

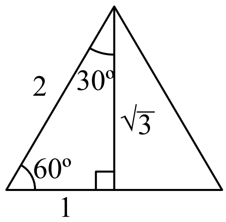


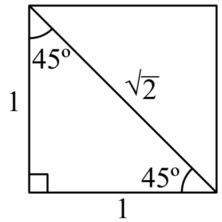
Reflex Angles (Teachers)
$\sin(180° + \alpha) = -\sin\alpha$
$\cos(180° + \alpha) = -\cos\alpha$
$\tan(180° + \alpha) = +\tan\alpha$

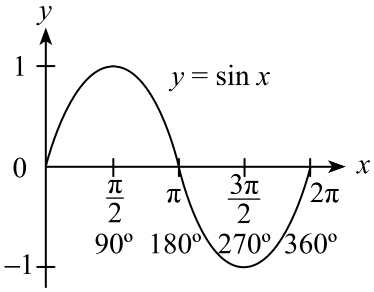
Reflex Angles (are Crazy)
$\sin(360° - \alpha) = \sin(-\alpha) = -\sin\alpha$
$\cos(360° - \alpha) = \cos(-\alpha) = +\cos\alpha$
$\tan(360° - \alpha) = \tan(-\alpha) = -\tan\alpha$

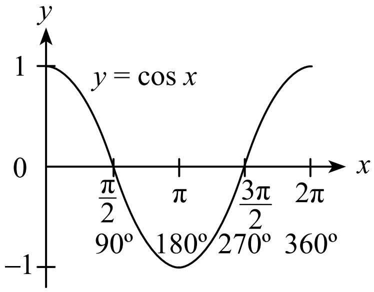
Trigonometric Ratios of Special/Common Angles

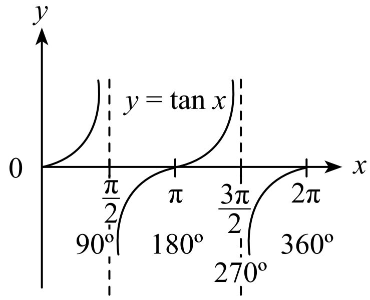
6











From the diagrams and graphs on page 115, the trigonometric ratios of special/common angles are obtained and summarised in the table below.

θ	0°	30°	45°	60°	90°	Pattern	180°	270°	360°
$\sin\theta$	$\frac{\sqrt{0}}{2}=0$	$\frac{\sqrt{1}}{2}=\frac{1}{2}$	$\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}=1$	$\frac{\sqrt{0}\rightarrow\sqrt{4}}{2}$	0	–1	0
$\cos\theta$	$\frac{\sqrt{4}}{2}=1$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$	$\frac{\sqrt{1}}{2}=\frac{1}{2}$	$\frac{\sqrt{0}}{2}=0$	$\frac{\sqrt{4}\rightarrow\sqrt{0}}{2}$	–1	0	–1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	NA	0	∞	0

Principal Values of an Angle

7 The principal values of an angle x is a chosen interval of x which corresponds to the interval $-1 \leqslant y \leqslant 1$.

Trigonometric function	Principal values
$y = \sin x$	$-90° \leqslant x \leqslant 90°$ or $-90° \leqslant \sin^{-1} y \leqslant 90°$
$y = \cos x$	$0° \leqslant x \leqslant 180°$ or $0° \leqslant \sin^{-1} y \leqslant 180°$
$y = \tan x$	$-90° < x < 90°$ or $-90° < \tan^{-1} y < 90°$

Angle Measure

8 Angles are measured in degrees or in radians.

9 The formulae for conversion are:

Converting angle from	Formulae
Radian to degree	$1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ}$
Degree to radian	$1° = \frac{\pi}{180} \text{ rad}$

10 Conversion table between degrees and radians for angles which are useful for trigonometric curve sketching:

Degree	22.5°	45°	67.5°	90°	112.5°	135°	157.5°	180°
Radian	$\frac{\pi}{8}$	$\frac{2\pi}{8}=\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{4\pi}{8}=\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{6\pi}{8}=\frac{3\pi}{4}$	$\frac{7\pi}{8}$	$\frac{8\pi}{8}=\pi$
Degree	202.5°	225°	247.5°	270°	292.5°	315°	337.5°	360°
Radian	$\frac{9\pi}{8}$	$\frac{10\pi}{8}=\frac{5\pi}{4}$	$\frac{11\pi}{8}$	$\frac{12\pi}{8}=\frac{3\pi}{2}$	$\frac{13\pi}{8}$	$\frac{14\pi}{8}=\frac{7\pi}{4}$	$\frac{15\pi}{8}$	$\frac{16\pi}{8}=2\pi$

Graphs of Basic Trigonometric Functions

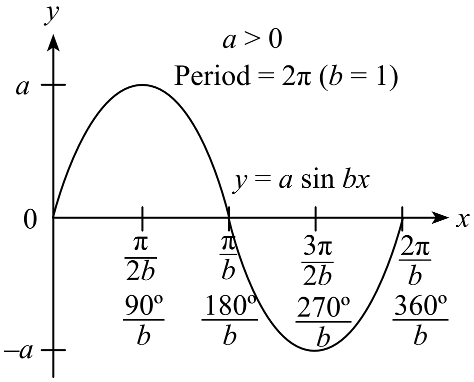
11 $y = a \sin bx + c$
$y = a \cos bx + c$
$y = a \tan bx$

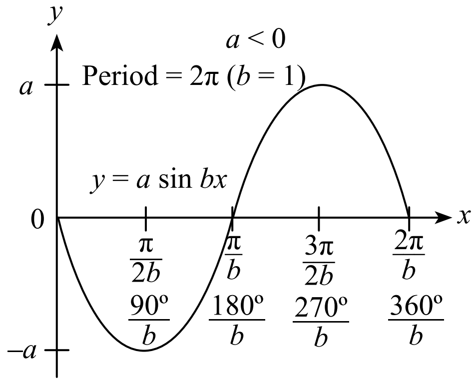
(a) a controls vertical stretch
For sine and cosine graphs, a is the amplitude

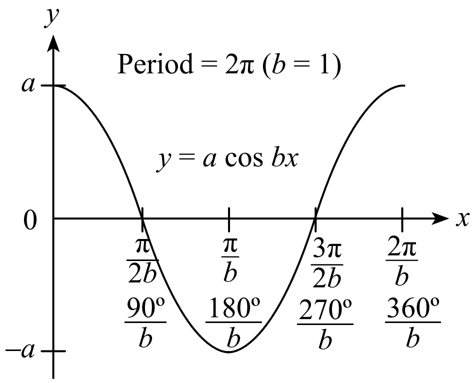
(b) b is the frequency
For sine and cosine graphs, period $= \frac{2\pi}{b}$ or $\frac{360°}{b}$

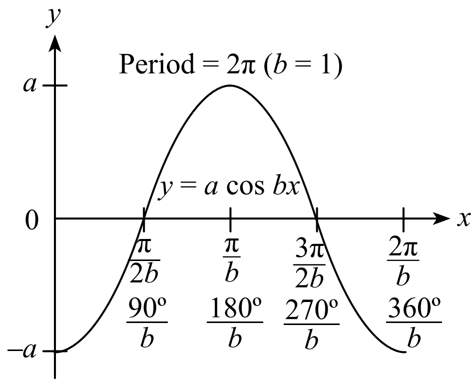
For tangent graphs, period $= \frac{\pi}{b}$ or $\frac{180°}{b}$

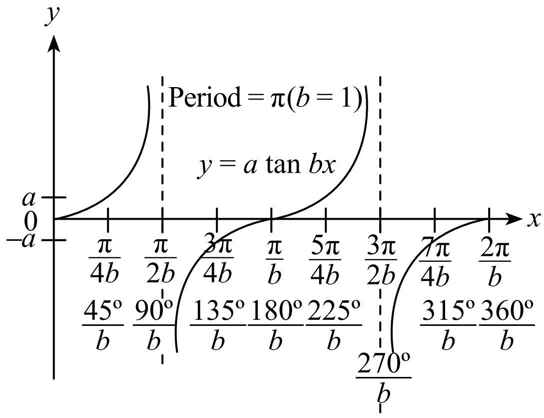
(c) c controls vertical translation
When c is positive, the graph shifts up c units
When c is negative, the graph shifts down c units

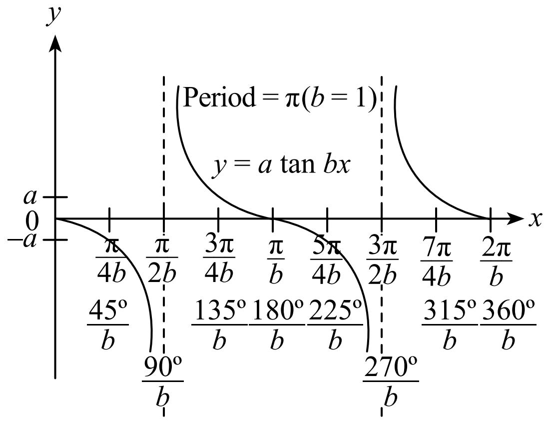












Fundamental Trigonometric Identities

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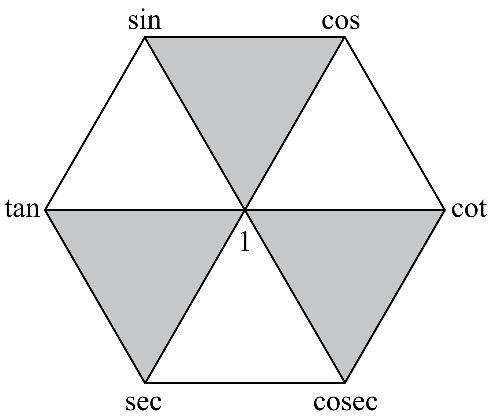

Ratio Identities	A trigonometric ratio is equal to the quotient of the next two trigonometric ratios taken in either clockwise or anti-clockwise direction.	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
		$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$
Reciprocal Identities	Trigonometric ratios at opposite ends of the hexagon are reciprocals of each other.	$\cot\theta = \dfrac{1}{\tan\theta}$
		$\sec\theta = \dfrac{1}{\cos\theta}$
		$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$
Pythagorean Identities	The sum of squares of two trigonometric ratios at the base of an inverted triangle is equal to the square of the trigonometric ratio at the vertex.	$\sin^2\theta + \cos^2\theta = 1$
		$\tan^2\theta + 1 = \sec^2\theta$
		$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$